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# Application of Poisson Integral Formula on Solving Some Definite Integrals

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**ABSTRACT**

This paper studies six types of definite integrals and uses Maple for verification. The closed forms of these definite integrals can be obtained mainly using Poisson integral formula. On the other hand, some examples are used to demonstrate the calculations.

***Keywords:*** *definite integrals, closed forms, Poisson integral formula, Maple.*

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## INTRODUCTION

In calculus and engineering mathematics, there are many methods to solve the integral problems including change of variables method, integration by parts method, partial fractions method, trigonometric substitution method, etc. In this paper, we study the following six types of definite integrals which are not easy to obtain their answers using the methods mentioned above.

$$\int_0^{2\pi} \frac{\exp(r\cos\theta)\cdot\cos(m\theta+r\sin\theta)}{r^2-2rs\cos(\theta-\varphi)+s^2}d\theta \quad , \quad (1)$$

$$\int_0^{2\pi} \frac{\exp(r\cos\theta)\cdot\sin(m\theta+r\sin\theta)}{r^2-2rs\cos(\theta-\varphi)+s^2}d\theta \quad , \quad (2)$$

$$\int_0^{2\pi} \frac{\cos m\theta\cdot\sin(r\cos\theta)\cosh(r\sin\theta)-\sin m\theta\cdot\cos(r\cos\theta)\sinh(r\sin\theta)}{r^2-2rs\cos(\theta-\varphi)+s^2}d\theta \quad , \quad (3)$$

$$\int_0^{2\pi} \frac{\sin m\theta\cdot\sin(r\cos\theta)\cosh(r\sin\theta)+\cos m\theta\cdot\cos(r\cos\theta)\sinh(r\sin\theta)}{r^2-2rs\cos(\theta-\varphi)+s^2}d\theta \quad , \quad (4)$$

$$\int_0^{2\pi} \frac{\cos m\theta\cdot\cos(r\cos\theta)\cosh(r\sin\theta)+\sin m\theta\cdot\sin(r\cos\theta)\sinh(r\sin\theta)}{r^2-2rs\cos(\theta-\varphi)+s^2}d\theta \quad , \quad (5)$$

$$\int_0^{2\pi} \frac{\sin m\theta\cdot\cos(r\cos\theta)\cosh(r\sin\theta)-\cos m\theta\cdot\sin(r\cos\theta)\sinh(r\sin\theta)}{r^2-2rs\cos(\theta-\varphi)+s^2}d\theta \quad , \quad (6)$$

3where $r, s, \varphi$ are real numbers, $|s| < |r|$, and $m$ is a non-negative integer. We can obtain the closed forms of these definite integrals mainly using Poisson integral formula; these are the major results of this paper (i.e., Theorems 1-3). Adams et al. [1], Nyblom [2], and Oster [3] provided some techniques to solve the integral problems. Yu [4-29], Yu and B. -H. Chen [30], and T. -J. Chen and Yu [31-33] used complex power series method, integration term by term theorem, differentiation with respect to a parameter, Parseval's theorem, and generalized Cauchy integral formula to solve some types of integrals. In this paper, some examples are used to demonstrate the proposed calculations, and the manual calculations are verified using Maple.

## *1. Main Results*

Some formulas used in this paper are introduced below.

### *2.1 Euler's formula:*

$e^{ix} = \cos x + i \sin x$, where $i = \sqrt{-1}$, and $x$ is any real number.

### *2.2 DeMoivre's formula:*

$(\cos x + i \sin x)^m = \cos mx + i \sin mx$, where $m$ is any integer, and $x$ is any real number.

The following two formulas can be found in [34, p62].

2.3 $\sin(a+ib) = \sin a \cosh b + i \cos a \sinh b$, where $a, b$ are real numbers.

2.4 $\cos(a+ib) = \cos a \cosh b - i \sin a \sinh b$, where $a, b$ are real numbers.

An important formula used in this study is introduced below, which can be found in [35, p 145].

### *2.5 Poisson integral formula:*

Suppose that $r, s$ are real numbers, and $|s| < |r|$. If $f$ is defined and continuous on the closed disc $\{z \in C \| z| \le |r|\}$ and is analytic on the open disc $\{z \in C \| z| < |r|\}$, then

$$f(se^{i\varphi}) = \frac{r^2 - s^2}{2\pi} \int_0^{2\pi} \frac{f(re^{i\theta})}{r^2 - 2rs\cos(\theta - \varphi) + s^2} d\theta.$$

In the following, we determine the closed forms of the definite integrals (1) and (2).

**Theorem 1** If $r, s, \varphi$ are real numbers, $|s| < |r|$, and $m$ is a non-negative integer, then the definite integrals

$$\int_0^{2\pi} \frac{\exp(r\cos\theta) \cdot \cos(m\theta + r\sin\theta)}{r^2 - 2rs\cos(\theta - \varphi) + s^2} d\theta = \frac{2\pi s^m}{r^m (r^2 - s^2)} \exp(s\cos\varphi) \cdot \cos(m\varphi + s\sin\varphi), \quad (7)$$

and

$$\int_0^{2\pi} \frac{\exp(r\cos\theta) \cdot \sin(m\theta + r\sin\theta)}{r^2 - 2rs\cos(\theta - \varphi) + s^2} d\theta = \frac{2\pi s^m}{r^m (r^2 - s^2)} \exp(s\cos\varphi) \cdot \sin(m\varphi + s\sin\varphi). \quad (8)$$

**Proof** Let $f(z) = z^m e^z$, then $f(z)$ is analytic on the whole complex plane. Using Poisson integral formula yields

$$(se^{i\varphi})^m \exp(se^{i\varphi}) = \frac{r^2 - s^2}{2\pi} \int_0^{2\pi} \frac{(re^{i\theta})^m \exp(re^{i\theta})}{r^2 - 2rs\cos(\theta - \varphi) + s^2} d\theta. \quad (9)$$

By Euler's formula and DeMoivre's formula, we have

$$s^m e^{im\varphi}\exp(se^{i\varphi}) = \frac{r^2-s^2}{2\pi}\int_0^{2\pi}\frac{r^m e^{im\theta}\exp(re^{i\theta})}{r^2-2rs\cos(\theta-\varphi)+s^2}d\theta \quad . \qquad (10)$$

Therefore,

$$\int_0^{2\pi}\frac{e^{im\theta}\exp(re^{i\theta})}{r^2-2rs\cos(\theta-\varphi)+s^2}d\theta = \frac{2\pi s^m}{r^m(r^2-s^2)}e^{im\varphi}\exp(se^{i\varphi}) \quad . \qquad (11)$$

Using the equality of real parts of both sides of Eq. (11) yields Eq. (7) holds. Also, by the equality of imaginary parts of both sides of Eq. (11), we obtain Eq. (8). q.e.d.

Next, the closed forms of the definite integrals (3) and (4) are obtained below.

**Theorem 2** If the assumptions are the same as Theorem 1, then

$$\int_0^{2\pi}\frac{\cos m\theta\cdot\sin(r\cos\theta)\cosh(r\sin\theta)-\sin m\theta\cdot\cos(r\cos\theta)\sinh(r\sin\theta)}{r^2-2rs\cos(\theta-\varphi)+s^2}d\theta$$

$$=\frac{2\pi s^m}{r^m(r^2-s^2)}[\cos m\varphi\cdot\sin(s\cos\varphi)\cosh(s\sin\varphi)-\sin m\varphi\cdot\cos(s\cos\varphi)\sinh(s\sin\varphi)] \quad , \qquad (12)$$

and

$$\int_0^{2\pi}\frac{\sin m\theta\cdot\sin(r\cos\theta)\cosh(r\sin\theta)+\cos m\theta\cdot\cos(r\cos\theta)\sinh(r\sin\theta)}{r^2-2rs\cos(\theta-\varphi)+s^2}d\theta$$

$$=\frac{2\pi s^m}{r^m(r^2-s^2)}[\sin m\varphi\cdot\sin(s\cos\varphi)\cosh(s\sin\varphi)+\cos m\varphi\cdot\cos(s\cos\varphi)\sinh(s\sin\varphi)] \quad . \qquad (13)$$

**Proof** Since $g(z)=z^m\sin z$ is analytic on the whole complex plane, using Poisson integral formula yields

$$(se^{i\varphi})^m\sin(se^{i\varphi}) = \frac{r^2-s^2}{2\pi}\int_0^{2\pi}\frac{(re^{i\theta})^m\sin(re^{i\theta})}{r^2-2rs\cos(\theta-\varphi)+s^2}d\theta \quad . \qquad (14)$$

It follows that

$$\int_0^{2\pi}\frac{e^{im\theta}\sin(re^{i\theta})}{r^2-2rs\cos(\theta-\varphi)+s^2}d\theta = \frac{2\pi s^m}{r^m(r^2-s^2)}e^{im\varphi}\sin(se^{i\varphi}) \quad . \qquad (15)$$

Eq. (12) can be obtained using Formula 2.3 and the equality of real parts of both sides of Eq. (15). On the other hand, by Formula 2.3 and the equality of imaginary parts of both sides of Eq. (15), we obtain Eq. (13). q.e.d.

Finally, we find the closed forms of the definite integrals (5) and (6).

**Theorem 3** If the assumptions are the same as Theorem 1, then

$$\int_0^{2\pi}\frac{\cos m\theta\cdot\cos(r\cos\theta)\cosh(r\sin\theta)+\sin m\theta\cdot\sin(r\cos\theta)\sinh(r\sin\theta)}{r^2-2rs\cos(\theta-\varphi)+s^2}d\theta$$

$$=\frac{2\pi s^m}{r^m(r^2-s^2)}[\cos m\varphi\cdot\cos(s\cos\varphi)\cosh(s\sin\varphi)+\sin m\varphi\cdot\sin(s\cos\varphi)\sinh(s\sin\varphi)] \quad , \qquad (16)$$

and

$$\int_0^{2\pi} \frac{\sin m\theta \cdot \cos(r\cos\theta)\cosh(r\sin\theta) - \cos m\theta \cdot \sin(r\cos\theta)\sinh(r\sin\theta)}{r^2 - 2rs\cos(\theta-\varphi) + s^2} d\theta$$

$$= \frac{2\pi s^m}{r^m(r^2-s^2)}[\sin m\varphi \cdot \cos(s\cos\varphi)\cosh(s\sin\varphi) - \cos m\varphi \cdot \sin(s\cos\varphi)\sinh(s\sin\varphi)] \quad (17)$$

**Proof** Since $h(z) = z^m \cos z$ is analytic on the whole complex plane, by Poisson integral formula and Formula 2.4, the desired results hold. q.e.d.

### *2. Examples*

In the following, for the six types of definite integrals in this study, some examples are proposed and we use Theorems 1-3 to determine their closed forms. On the other hand, Maple is used to calculate the approximations of some definite integrals and their solutions for verifying our answers.

***Example 1*** In Eq. (7), if $r = 4, s = 2, \varphi = \pi/3$, and $m = 3$, then

$$\int_0^{2\pi} \frac{\exp(4\cos\theta)\cdot\cos(3\theta + 4\sin\theta)}{20 - 16\cos(\theta - \pi/3)} d\theta = \frac{\pi}{48}\exp(1)\cdot\cos(\pi + \sqrt{3}) \quad (18)$$

Next, we use Maple to verify the correctness of Eq. (18).

```
>evalf(int(exp(4*cos(theta))*cos(3*theta+4*sin(theta))/(20-16*cos(theta-Pi/3)),theta=0..2*Pi),20);
0.028564795148756130217
>evalf(Pi/48*exp(1)*cos(Pi+sqrt(3)),20);
0.028564795148756130214
```

On the other hand, if $r = 5, s = 4, \varphi = \pi/4$, and $m = 6$ in Eq. (8), then

$$\int_0^{2\pi} \frac{\exp(5\cos\theta)\cdot\sin(6\theta + 5\sin\theta)}{41 - 40\cos(\theta - \pi/4)} d\theta = \frac{8192\pi}{140625}\exp(2\sqrt{2})\cdot\sin(3\pi/2 + 2\sqrt{2}) \quad (19)$$

```
>evalf(int(exp(5*cos(theta))*sin(6*theta+5*sin(theta))/(41-40*cos(theta-Pi/4)),theta=0..2*Pi),20);
2.9457364531215630498
>evalf(8192*Pi/140625*exp(2*sqrt(2))*sin(3*Pi/2+2*sqrt(2)),20);
2.9457364531215630497
```

**Example 2** In Eq. (12), let $r = 7, s = 5, \varphi = \pi/6$, and $m = 4$, then the definite integral

$$\int_0^{2\pi} \frac{\cos 4\theta \cdot \sin(7\cos\theta)\cosh(7\sin\theta) - \sin 4\theta \cdot \cos(7\cos\theta)\sinh(7\sin\theta)}{74 - 70\cos(\theta - \pi/6)} d\theta$$

$$= \frac{625\pi}{28812}[-1/2\cdot\sin(5\sqrt{3}/2)\cosh(5/2) - \sqrt{3}/2\cdot\cos(5\sqrt{3}/2)\sinh(5/2)] \quad (20)$$

We also use Maple to verify the correctness of Eq. (20).

```
>evalf(int((cos(4*theta)*sin(7*cos(theta))*cosh(7*sin(theta))-sin(4*theta)*cos(7*cos(theta))*sinh(7*sin(theta)))/(74-
70*cos(theta-Pi/6)),theta=0..2*Pi),20);
0.32706714869219491572
>evalf(625*Pi/28812*(-1/2*sin(5*sqrt(3)/2)*cosh(5/2)-sqrt(3)/2*cos(5*sqrt(3)/2)*sinh(5/2)),20);
0.32706714869219491570
```

Also, if $r = -3, s = 2, \varphi = \pi/3$, and $m = 2$ in Eq. (13), then

$$\int_0^{2\pi} \frac{\sin 2\theta \cdot \sin(-3\cos\theta)\cosh(-3\sin\theta) + \cos 2\theta \cdot \cos(-3\cos\theta)\sinh(-3\sin\theta)}{13+12\cos(\theta-\pi/3)} d\theta$$
$$= \frac{8\pi}{45}[\sqrt{3}/2 \cdot \sin(1)\cosh(\sqrt{3}) - 1/2 \cdot \cos(1)\sinh(\sqrt{3})] . \quad (21)$$

```
>evalf(int((sin(2*theta)*sin(-3*cos(theta))*cosh(-3*sin(theta))+cos(2*theta)*cos(-3*cos(
theta))*sinh(-3*sin(theta)))/(13+12*cos(theta-Pi/3)),theta=0..2*Pi),20);
0.77318048433575699712
>evalf(8*Pi/45*(sqrt(3)/2*sin(1)*cosh(sqrt(3))-1/2*cos(1)*sinh(sqrt(3))),20);
0.77318048433575699713
```

**Example 3** In Eq. (16), if $r=4, s=-2, \varphi=2\pi/3$, and $m=5$, then the definite integral

$$\int_0^{2\pi} \frac{\cos 5\theta \cdot \cos(4\cos\theta)\cosh(4\sin\theta) + \sin 5\theta \cdot \sin(4\cos\theta)\sinh(4\sin\theta)}{20+16\cos(\theta-2\pi/3)} d\theta$$
$$= \frac{-\pi}{192}[-1/2 \cdot \cos(1)\cosh(-\sqrt{3}) - \sqrt{3}/2 \cdot \sin(1)\sinh(-\sqrt{3})] . \quad (22)$$

Maple is used to verify the correctness of Eq. (22) as follows:

```
>evalf(int((cos(5*theta)*cos(4*cos(theta))*cosh(4*sin(theta))+sin(5*theta)*sin(4*cos(theta))*sinh(4*sin(theta)))/(20+16*cos(t
heta-2*Pi/3)),theta=0..2*Pi),15);
-0.0197601292496753
>evalf(-Pi/192*(-1/2*cos(1)*cosh(-sqrt(3))-sqrt(3)/2*sin(1)*sinh(-sqrt(3))),15);
-0.0197601292496755
```

In addition, let $r=-5, s=4, \varphi=-3\pi/4$, and $m=7$ in Eq. (17), we obtain

$$\int_0^{2\pi} \frac{\sin 7\theta \cdot \cos(-5\cos\theta)\cosh(-5\sin\theta) - \cos 7\theta \cdot \sin(-5\cos\theta)\sinh(-5\sin\theta)}{41+40\cos(\theta+3\pi/4)} d\theta$$
$$= \frac{32768\pi}{-703125}[\sqrt{2}/2 \cdot \cos(-2\sqrt{2})\cosh(-2\sqrt{2}) + \sqrt{2}/2 \cdot \sin(-2\sqrt{2})\sinh(-2\sqrt{2})] . \quad (23)$$

```
>evalf(int((sin(7*theta)*cos(-5*cos(theta))*cosh(-5*sin(theta))-cos(7*theta)*sin(-5*cos(
theta))*sinh(-5*sin(theta)))/(41+40*cos(theta+3*Pi/4)),theta=0..2*Pi),18);
0.567231798371978247
>evalf(32768*Pi/(-703125)*(sqrt(2)/2*cos(-2*sqrt(2))*cosh(-2*sqrt(2))+sqrt(2)/2*sin(-2*
sqrt(2))*sinh(-2*sqrt(2))),18);
0.567231798371978248
```

### *4. Conclusion*

In this article, we use Poisson integral formula to solve some types of definite integrals. In fact, the applications of this formula are extensive, and can be used to easily solve many difficult problems; we endeavor to conduct further studies on related applications. In addition, Maple also plays a vital assistive role in problem-solving. In the future, we will extend the research topic to other calculus and engineering mathematics problems and use Maple to verify our answers.

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